

Due:
Grace period until

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Continuous Intro

(a) Is

$$f(x) = \begin{cases} 2x, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

a valid density function? Why or why not? Is it a valid CDF? Why or why not?

(b) Calculate the PDF $f_X(x)$, along with $\mathbb{E}[X]$ and $\text{Var}(X)$ if the CDF of X is

$$F_X(x) = \begin{cases} 0, & x \leq 0 \\ \frac{x}{\ell}, & 0 \leq x \leq \ell, \\ 1, & x \geq \ell \end{cases}$$

(c) Suppose X and Y are independent and have densities

$$f_X(x) = \begin{cases} 2x, & 0 \leq x \leq 1, \\ 0, & \text{otherwise,} \end{cases} \quad f_Y(y) = \begin{cases} 1, & 0 \leq y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

What is their joint distribution? (Hint: for parts (c) and (d), we can use independence in much the same way that we did in discrete probability)

(d) Calculate $\mathbb{E}[XY]$ for the X and Y in part (c).

2 Lunch Meeting

Alice and Bob agree to try to meet for lunch between 12 PM and 1 PM at their favorite sushi restaurant. Being extremely busy, they are unable to specify their arrival times exactly, and can say only that each of them will arrive (independently) at a time that is uniformly distributed within the hour. In order to avoid wasting precious time, if the other person is not there when they arrive they agree to wait exactly fifteen minutes before leaving. What is the probability that they will actually meet for lunch? (hint: Sketch the joint distribution of the arrival times of Alice and Bob. What parts of the distribution corresponds to them meeting for lunch?)

3 Darts with Friends

Michelle and Alex are playing darts. Being the better player, Michelle's aim follows a uniform distribution over a disk of radius 1 around the center. Alex's aim follows a uniform distribution over a disk of radius 2 around the center.

- (a) Let the distance of Michelle's throw from the center be denoted by the random variable X and let the distance of Alex's throw from the center be denoted by the random variable Y .
 - What's the cumulative distribution function of X ?
 - What's the cumulative distribution function of Y ?
 - What's the probability density function of X ?
 - What's the probability density function of Y ?
- (b) What's the probability that Michelle's throw is closer to the center than Alex's throw? What's the probability that Alex's throw is closer to the center?
- (c) What's the cumulative distribution function of $U = \max\{X, Y\}$?
- (d) What's the cumulative distribution function of $V = \min\{X, Y\}$?
- (e) What is the expectation of the absolute difference between Michelle's and Alex's distances from the center, that is, what is $\mathbb{E}[|X - Y|]$? [Hint: Use parts (c) and (d), together with the continuous version of the tail sum formula, which states that $\mathbb{E}[Z] = \int_0^\infty \mathbb{P}[Z \geq z] dz$.]

4 Waiting For the Bus

Edward and Jerry are waiting at the bus stop outside of Soda Hall.

Like many bus systems, buses arrive in periodic intervals. However, the Berkeley bus system is unreliable, so the length of these intervals are random, and follow Exponential distributions.

Edward is waiting for the 51B, which arrives according to an Exponential distribution with parameter λ . That is, if we let the random variable X_i correspond to the difference between the arrival time i th and $(i - 1)$ st bus (also known as the inter-arrival time) of the 51B, $X_i \sim \text{Expo}(\lambda)$.

Jerry is waiting for the 79, whose inter-arrival times also follows Exponential distributions with parameter μ . That is, if we let Y_i denote the inter-arrival time of the 79, $Y_i \sim \text{Expo}(\mu)$. Assume that all inter-arrival times are independent.

- (a) What is the probability that Jerry's bus arrives before Edward's bus?
- (b) After 20 minutes, the 79 arrives, and Jerry rides the bus. However, the 51B still hasn't arrived yet. Let D be the additional amount of time Edward needs to wait for the 51B to arrive. What is the distribution of D ?
- (c) Lavanya isn't picky, so she will wait until either the 51B or the 79 bus arrives. Find the distribution of Z , the amount of time Lavanya will wait before catching her bus.
- (d) Khalil doesn't feel like riding the bus with Edward. He decides that he will wait for the second arrival of the 51B to ride the bus. Find the distribution of $T = X_1 + X_2$, the amount of time that Khalil will wait to ride the bus.

5 Chebyshev's Inequality vs. Central Limit Theorem

Let n be a positive integer. Let X_1, X_2, \dots, X_n be i.i.d. random variables with the following distribution:

$$\mathbb{P}[X_i = -1] = \frac{1}{12}; \quad \mathbb{P}[X_i = 1] = \frac{9}{12}; \quad \mathbb{P}[X_i = 2] = \frac{2}{12}.$$

- (a) Calculate the expectations and variances of X_1 , $\sum_{i=1}^n X_i$, $\sum_{i=1}^n (X_i - \mathbb{E}[X_i])$, and

$$Z_n = \frac{\sum_{i=1}^n (X_i - \mathbb{E}[X_i])}{\sqrt{n/2}}.$$

- (b) Use Chebyshev's Inequality to find an upper bound b for $\mathbb{P}[|Z_n| \geq 2]$.
- (c) Can you use b to bound $\mathbb{P}[Z_n \geq 2]$ and $\mathbb{P}[Z_n \leq -2]$?
- (d) As $n \rightarrow \infty$, what is the distribution of Z_n ?
- (e) We know that if $Z \sim \mathcal{N}(0, 1)$, then $\mathbb{P}[|Z| \leq 2] = \Phi(2) - \Phi(-2) \approx 0.9545$. As $n \rightarrow \infty$, can you provide approximations for $\mathbb{P}[Z_n \geq 2]$ and $\mathbb{P}[Z_n \leq -2]$?