

Due: Saturday, 10/29, 4:00 PM
Grace period until Saturday, 10/29, 6:00 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Cliques in Random Graphs

Consider the graph $G = (V, E)$ on n vertices which is generated by the following random process: for each pair of vertices u and v , we flip a fair coin and place an (undirected) edge between u and v if and only if the coin comes up heads.

- (a) What is the size of the sample space?
- (b) A k -clique in a graph is a set S of k vertices which are pairwise adjacent (every pair of vertices is connected by an edge). For example, a 3-clique is a triangle. Let E_S be the event that a set S forms a clique. What is the probability of E_S for a particular set S of k vertices?
- (c) Suppose that $V_1 = \{v_1, \dots, v_\ell\}$ and $V_2 = \{w_1, \dots, w_k\}$ are two arbitrary sets of vertices. What conditions must V_1 and V_2 satisfy in order for E_{V_1} and E_{V_2} to be independent? Prove your answer.
- (d) Prove that $\binom{n}{k} \leq n^k$. (You might find this useful in part (e))
- (e) Prove that the probability that the graph contains a k -clique, for $k \geq 4\log_2 n + 1$, is at most $1/n$.

2 Symmetric Marbles

A bag contains 4 red marbles and 4 blue marbles. Leanne and Sylvia play a game where they draw four marbles in total, one by one, uniformly at random, without replacement. Leanne wins if there are more red than blue marbles, and Sylvia wins if there are more blue than red marbles. If there are an equal number of marbles, the game is tied.

- (a) Let A_1 be the event that the first marble is red and let A_2 be the event that the second marble is red. Are A_1 and A_2 independent?

- (b) What is the probability that Leanne wins the game?
- (c) Given that Leanne wins the game, what is the probability that all of the marbles were red?

Now, suppose the bag contains 8 red marbles and 4 blue marbles. Moreover, if there are an equal number of red and blue marbles among the four drawn, Leanne wins if the third marble is red, and Sylvia wins if the third marble is blue.

- (d) What is the probability that the third marble is red?
- (e) Given that there are k red marbles among the four drawn, where $0 \leq k \leq 4$, what is the probability that the third marble is red? Answer in terms of k .
- (f) Given that the third marble is red, what is the probability that Leanne wins the game?

3 PIE Extended

One interesting result yielded by the Principle of Inclusion and Exclusion (PIE) is that for any events $A_1, A_2, \dots, A_n$ in some probability space,

$$\sum_{i=1}^n \mathbb{P}[A_i] - \sum_{i < j \leq n} \mathbb{P}[A_i \cap A_j] + \sum_{i < j < k \leq n} \mathbb{P}[A_i \cap A_j \cap A_k] - \dots + (-1)^{n-1} \mathbb{P}[A_1 \cap A_2 \cap \dots \cap A_n] \geq 0$$

(Note the LHS is equal to $\mathbb{P}[\bigcup_{i=1}^n A_i]$ by PIE, and probability is nonnegative).

Prove that for any events $A_1, A_2, \dots, A_n$ in some probability space,

$$\sum_{i=1}^n \mathbb{P}[A_i] - 2 \sum_{i < j \leq n} \mathbb{P}[A_i \cap A_j] + 4 \sum_{i < j < k \leq n} \mathbb{P}[A_i \cap A_j \cap A_k] - \dots + (-2)^{n-1} \mathbb{P}[A_1 \cap A_2 \cap \dots \cap A_n] \geq 0$$

(Hint: consider defining an event B to represent "an odd number of $A_1, \dots, A_n$ occur")

4 Independent Complements

Let Ω be a sample space, and let $A, B \subseteq \Omega$ be two independent events.

- (a) Prove or disprove: $\bar{A}$ and $\bar{B}$ must be independent.
- (b) Prove or disprove: A and $\bar{B}$ must be independent.
- (c) Prove or disprove: A and $\bar{A}$ must be independent.
- (d) Prove or disprove: It is possible that $A = B$.

5 Monty Hall's Revenge

Due to a quirk of the television studio's recruitment process, Monty Hall has ended up drawing all the contestants for his game show from among the ranks of former CS70 students. Unfortunately for Monty, the former students' amazing probability skills have made his cars-and-goats gimmick unprofitable for the studio. Monty decides to up the stakes by asking his contestants to generalise to three new situations with a variable number of doors, goats, and cars:

- (a) There are n doors for some $n > 2$. One has a car behind it, and the remaining $n - 1$ have goats. As in the ordinary Monty Hall problem, Monty will reveal one door with a goat behind it after you make your first selection. How would switching affect the odds that you select the car? (Compute the probability of winning in both scenarios, and compare the results.)
(Hint: Think about the size of the sample space for the experiment where you *always* switch. How many of those outcomes are favorable?)
- (b) Again there are $n > 2$ doors, one with a car and $n - 1$ with goats, but this time Monty will reveal $n - 2$ doors with goats behind them instead of just one. How does switching affect the odds of winning in this modified scenario?
- (c) Finally, imagine there are $k < n - 1$ cars and $n - k$ goats behind the $n > 2$ doors. After you make your first pick, Monty will reveal $j < n - k$ doors with goats. What values of j, k maximize the relative improvement in your odds of winning if you choose to switch? (i.e. what j, k maximizes the ratio between your odds of winning when you switch, and your odds of winning when you do not switch?)

6 (Un)conditional (In)equalities

Let us consider a sample space $\Omega = \{\omega_1, \dots, \omega_N\}$ of size $N > 2$ and two probability functions $\mathbb{P}_1$ and $\mathbb{P}_2$ on it. That is, we have two probability spaces: $(\Omega, \mathbb{P}_1)$ and $(\Omega, \mathbb{P}_2)$.

- (a) Suppose that for every subset $A \subseteq \Omega$ of size $|A| = 2$ and for every outcome $\omega \in \Omega$, it is true that $\mathbb{P}_1[\omega | A] = \mathbb{P}_2[\omega | A]$.
 - (i) Let $A = \{\omega_i, \omega_j\}$ for some $i, j \in \{1, \dots, N\}$. What is the relationship between $\frac{\mathbb{P}_1[\omega_i]}{\mathbb{P}_1[\omega_j]}$ and $\frac{\mathbb{P}_2[\omega_i]}{\mathbb{P}_2[\omega_j]}$?
 - (ii) Is it necessarily true that $\mathbb{P}_1[\omega] = \mathbb{P}_2[\omega]$ for all $\omega \in \Omega$? That is, if $\mathbb{P}_1$ and $\mathbb{P}_2$ are equal conditional on events of size 2, are they equal unconditionally? (*Hint*: Remember that probabilities must add up to 1.)
- (b) Suppose that for every subset $A \subseteq \Omega$ of size $|A| = k$, where k is some fixed element in $\{2, \dots, N\}$, and for every outcome $\omega \in \Omega$, it is true that $\mathbb{P}_1[\omega | A] = \mathbb{P}_2[\omega | A]$. Is it necessarily true that $\mathbb{P}_1[\omega] = \mathbb{P}_2[\omega]$ for all $\omega \in \Omega$? (*Hint*: Use part (a).)

For the following two parts, assume that $\Omega = \{(a_1, \dots, a_k) \mid \sum_{j=1}^k a_j = n\}$ is the set of configurations of n balls into k labeled bins, and let $\mathbb{P}_1$ be the probabilities assigned to these configurations by throwing the balls independently one after another and they will land into any of the k bins uniformly at random, and let $\mathbb{P}_2$ be the probabilities assigned to these configurations by uniformly sampling one of these configurations.

As an example, suppose $k = 6$ and $n = 2$. $\mathbb{P}_1$ is equivalent to rolling 2 six-sided dice, and letting a_i be the number of i 's that appear. $\mathbb{P}_2$ is equivalent to sampling uniformly from all unordered pairs (i, j) with $1 \leq i, j \leq 6$.

(c) Let A be the event that all n balls are in exactly one bin.

- (i) What are $\mathbb{P}_1[\omega \mid A]$ and $\mathbb{P}_2[\omega \mid A]$ for any $\omega \in A$?
 - (ii) Repeat part (i) for $\omega \in \Omega \setminus A$.
 - (iii) Is it true that $\mathbb{P}_1[\omega] = \mathbb{P}_2[\omega]$ for all $\omega \in \Omega$?
- (d) For the special case of $n = 9$ and $k = 3$, provide two outcomes B and C , so that $\mathbb{P}_1[B] < \mathbb{P}_2[B]$ and $\mathbb{P}_1[C] > \mathbb{P}_2[C]$. Provide justification.