

Due: Saturday, 9/17, 4:00 PM  
Grace period until Saturday, 9/17, 6:00 PM

## Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

## 1 Build-Up Error?

What is wrong with the following "proof"? In addition to finding a counterexample, you should explain what is fundamentally wrong with this approach, and why it demonstrates the danger of build-up error.

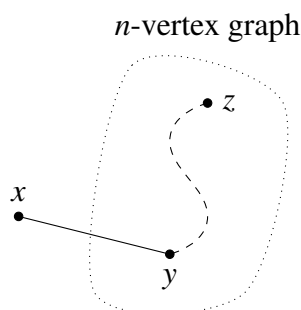
**False Claim:** If every vertex in an undirected graph has degree at least 1, then the graph is connected.

*Proof?* We use induction on the number of vertices  $n \geq 1$ .

*Base case:* There is only one graph with a single vertex and it has degree 0. Therefore, the base case is vacuously true, since the if-part is false.

*Inductive hypothesis:* Assume the claim is true for some  $n \geq 1$ .

*Inductive step:* We prove the claim is also true for  $n + 1$ . Consider an undirected graph on  $n$  vertices in which every vertex has degree at least 1. By the inductive hypothesis, this graph is connected. Now add one more vertex  $x$  to obtain a graph on  $(n + 1)$  vertices, as shown below.



All that remains is to check that there is a path from  $x$  to every other vertex  $z$ . Since  $x$  has degree at least 1, there is an edge from  $x$  to some other vertex; call it  $y$ . Thus, we can obtain a path from  $x$  to  $z$  by adjoining the edge  $\{x, y\}$  to the path from  $y$  to  $z$ . This proves the claim for  $n + 1$ .  $\square$

## 2 Tournament

A *tournament* is defined to be a directed graph such that for every pair of distinct nodes  $v$  and  $w$ , exactly one of  $(v, w)$  and  $(w, v)$  is an edge (representing which player beat the other in a round-robin tournament).

Prove that every tournament has a Hamiltonian path. In other words, you can always arrange the players in a line so that each player beats the next player in the line.

## 3 Proofs in Graphs

- (a) On the axis from San Francisco traffic habits to Los Angeles traffic habits, Old California is more towards San Francisco: that is, civilized. In Old California, all roads were one way streets. Suppose Old California had  $n$  cities ( $n \geq 2$ ) such that for every pair of cities  $X$  and  $Y$ , either  $X$  had a road to  $Y$  or  $Y$  had a road to  $X$ .

Prove that there existed a city which was reachable from every other city by traveling through at most 2 roads.

[Hint: Induction]

- (b) Consider a connected graph  $G$  with  $n$  vertices which has exactly  $2m$  vertices of odd degree, where  $m > 0$ . Prove that there are  $m$  walks that *together* cover all the edges of  $G$  (i.e., each edge of  $G$  occurs in exactly one of the  $m$  walks, and each of the walks should not contain any particular edge more than once).

[Hint: In lecture, we have shown that a connected undirected graph has an Eulerian tour if and only if every vertex has even degree. This fact may be useful in the proof.]

- (c) Prove that any graph  $G$  is bipartite if and only if it has no tours of odd length.

[Hint: In one of the directions, consider the lengths of paths starting from a given vertex.]

## 4 Planarity and Graph Complements

Let  $G = (V, E)$  be an undirected graph. We define the complement of  $G$  as  $\overline{G} = (V, \overline{E})$  where  $\overline{E} = \{(i, j) \mid i, j \in V, i \neq j\} - E$ ; that is,  $\overline{G}$  has the same set of vertices as  $G$ , but an edge  $e$  exists in  $\overline{G}$  if and only if it does not exist in  $G$ .

- (a) Suppose  $G$  has  $v$  vertices and  $e$  edges. How many edges does  $\overline{G}$  have?
- (b) Prove that for any graph with at least 13 vertices,  $G$  being planar implies that  $\overline{G}$  is non-planar.

- (c) Now consider the converse of the previous part, i.e., for any graph  $G$  with at least 13 vertices, if  $\overline{G}$  is non-planar, then  $G$  is planar. Construct a counterexample to show that the converse does not hold.

*Hint: Recall that if a graph contains a copy of  $K_5$ , then it is non-planar. Can this fact be used to construct a counterexample?*

## 5 Touring Hypercube

In the lecture, you have seen that if  $G$  is a hypercube of dimension  $n$ , then

- The vertices of  $G$  are the binary strings of length  $n$ .
- $u$  and  $v$  are connected by an edge if they differ in exactly one bit location.

A *Hamiltonian tour* of a graph is a sequence of vertices  $v_0, v_1, \dots, v_k$  such that:

- Each vertex appears exactly once in the sequence.
- Each pair of consecutive vertices is connected by an edge.
- $v_0$  and  $v_k$  are connected by an edge.

- (a) Show that a hypercube has an Eulerian tour if and only if  $n$  is even.
- (b) Show that every hypercube has a Hamiltonian tour.

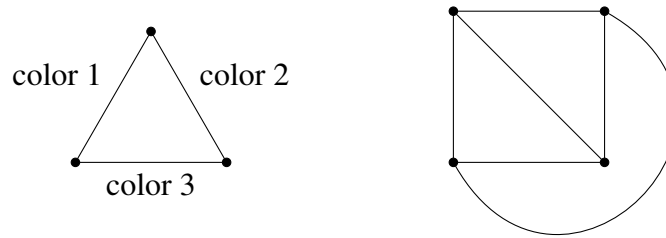
## 6 Connectivity

Consider the following claims regarding connectivity:

- (a) Prove: If  $G$  is a graph with  $n$  vertices such that for any two non-adjacent vertices  $u$  and  $v$ , it holds that  $\deg u + \deg v \geq n - 1$ , then  $G$  is connected.
- [*Hint: Show something more specific: for any two non-adjacent vertices  $u$  and  $v$ , there must be a vertex  $w$  such that  $u$  and  $v$  are both adjacent to  $w$ .*]
- (b) Give an example to show that if the condition  $\deg u + \deg v \geq n - 1$  is replaced with  $\deg u + \deg v \geq n - 2$ , then  $G$  is not necessarily connected.
- (c) Prove: For a graph  $G$  with  $n$  vertices, if the degree of each vertex is at least  $n/2$ , then  $G$  is connected.
- (d) Prove: If there are exactly two vertices with odd degrees in a graph, then they must be in the same connected component (meaning, there is a path connecting these two vertices).
- [*Hint: Proof by contradiction.*]

## 7 Edge Colorings

An edge coloring of a graph is an assignment of colors to edges in a graph where any two edges incident to the same vertex have different colors. An example is shown on the left.



- (a) Show that the 4 vertex complete graph above can be 3 edge colored. (Use the numbers 1, 2, 3 for colors. A figure is shown on the right.)
- (b) Prove that any graph with maximum degree  $d \geq 1$  can be edge colored with  $2d - 1$  colors.
- (c) Show that a tree can be edge colored with  $d$  colors where  $d$  is the maximum degree of any vertex.