

Due: Saturday, 9/3, 4:00 PM
Grace period until Saturday, 9/3, 6:00 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Solving a System of Equations

Alice wants to buy apples, beets, and carrots. An apple, a beet, and a carrot cost 16 dollars, two apples and three beets cost 23 dollars, and one apple, two beets, and three carrots cost 35 dollars. What are the prices for an apple, for a beet, and for a carrot, respectively? Set up a system of equations and show your work.

2 Calculus Review

(a) Compute the following integral:

$$\int_0^{\infty} \sin(t)e^{-t} dt.$$

(b) Find the minimum value of the following function over the reals and determine where it occurs.

$$f(x) = \int_0^{x^2} e^{-t^2} dt.$$

Show your work.

(c) Compute the double integral

$$\iint_R 2x + y dA,$$

where R is the region bounded by the lines $x = 1$, $y = 0$, and $y = x$.

3 Implication

Which of the following assertions are true no matter what proposition Q represents? For any false assertion, state a counterexample (i.e. come up with a statement $Q(x,y)$ that would make the implication false). For any true assertion, give a brief explanation for why it is true.

- (a) $\exists x \exists y Q(x, y) \implies \exists y \exists x Q(x, y)$.
- (b) $\forall x \exists y Q(x, y) \implies \exists y \forall x Q(x, y)$.
- (c) $\exists x \forall y Q(x, y) \implies \forall y \exists x Q(x, y)$.
- (d) $\exists x \exists y Q(x, y) \implies \forall y \exists x Q(x, y)$.

4 Logical Equivalence?

Decide whether each of the following logical equivalences is correct and justify your answer.

- (a) $\forall x (P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$
- (b) $\forall x (P(x) \vee Q(x)) \equiv \forall x P(x) \vee \forall x Q(x)$
- (c) $\exists x (P(x) \vee Q(x)) \equiv \exists x P(x) \vee \exists x Q(x)$
- (d) $\exists x (P(x) \wedge Q(x)) \equiv \exists x P(x) \wedge \exists x Q(x)$

5 Preserving Set Operations

For a function f , define the image of a set X to be the set $f(X) = \{y \mid y = f(x) \text{ for some } x \in X\}$. Define the inverse image or preimage of a set Y to be the set $f^{-1}(Y) = \{x \mid f(x) \in Y\}$. Prove the following statements, in which A and B are sets. By doing so, you will show that inverse images preserve set operations, but images typically do not.

Recall: For sets X and Y , $X = Y$ if and only if $X \subseteq Y$ and $Y \subseteq X$. To prove that $X \subseteq Y$, it is sufficient to show that $(\forall x) ((x \in X) \implies (x \in Y))$.

- (a) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.
- (b) $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$.
- (c) $f(A \cap B) \subseteq f(A) \cap f(B)$, and give an example where equality does not hold.
- (d) $f(A \setminus B) \supseteq f(A) \setminus f(B)$, and give an example where equality does not hold.

6 Prove or Disprove

For each of the following, either prove the statement, or disprove by finding a counterexample.

- (a) $(\forall n \in \mathbb{N})$ if n is odd then $n^2 + 4n$ is odd.
- (b) $(\forall a, b \in \mathbb{R})$ if $a + b \leq 15$ then $a \leq 11$ or $b \leq 4$.
- (c) $(\forall r \in \mathbb{R})$ if r^2 is irrational, then r is irrational.
- (d) $(\forall n \in \mathbb{Z}^+) 5n^3 > n!$. (Note: $\mathbb{Z}^+$ is the set of positive integers)

7 Rationals and Irrationals

Prove that the product of a non-zero rational number and an irrational number is irrational.

8 Twin Primes

- (a) Let $p > 3$ be a prime. Prove that p is of the form $3k + 1$ or $3k - 1$ for some integer k .
- (b) *Twin primes* are pairs of prime numbers p and q that have a difference of 2. Use part (a) to prove that 5 is the only prime number that takes part in two different twin prime pairs.