CS 70 Fall 2022 | Discrete Mathematics and Probability Theory
Satish Rao and Babak Ayazifar | DIS 14B

1 Joint Practice

Suppose that X and Y are random variables with joint density

$$f_{X,Y}(x,y) = \begin{cases} Ax^2y^2 & \text{if } 0 \le x \le 1 \text{ and } 0 \le y \le 1, \\ 0 & \text{otherwise,} \end{cases}$$

where A is a positive constant.

(a) What is the value of A?

(b) What is the marginal density of X?

(c) What is $\text{cov}(X,Y)$?

2 Sampling a Gaussian With Uniform

In this question, we will see one way to generate a normal random variable if we have access to a random number generator that outputs numbers between 0 and 1 uniformly at random.

As a general comment, remember that showing two random variables have the same CDF or PDF is sufficient for showing that they have the same distribution.

(a) First, let us see how to generate an exponential random variable with a uniform random variable. Let $U_1 \sim Uniform(0,1)$. Prove that $-\ln U_1 \sim Expo(1)$.

(b) Let $N_1, N_2 \sim \mathcal{N}(0,1)$, where N_1 and N_2 are independent. Prove that $N_1^2 + N_2^2 \sim Expo(1/2)$.

Hint: You may use the fact that over a region R, if we convert to polar coordinates $(x,y) \to (r,\theta)$, then the double integral over the region R will be

$$\iint_R f(x,y)\,dx\,dy = \iint_R f(r\cos\theta, r\sin\theta) \cdot r\,dr\,d\theta.$$

(c) Suppose we have two uniform random variables, U_1 and U_2. How would you transform these two random variables into a normal random variable with mean 0 and variance 1?

Hint: What part (b) tells us is that the point (N_1, N_2) where $N_1, N_2 \sim \mathcal{N}(0,1)$ will have a distance from the origin that is distributed as the square root of an exponential distribution. Thus to generate (N_1, N_2) use U_1 to sample the radius, and then use U_2 to sample the angle, and then you have two samples N_1 and N_2 from the $\mathcal{N}(0,1)$.

3 Suspicious Envelopes

There are two sealed envelopes. One containing x dollars and the other one containing $2x$ dollars. You select one of the two envelopes at random (but, don't open it).

(a) According to the logic below, you should keep swapping the selected envelope with the other one indefinitely to improve your expected earning. Is something wrong in this logic?

Logic: Let F and S denote the the amount in the envelope you select at random and the other one, respectively.Then, $S = 2F$ or $\frac{F}{2}$, with equal probability $\frac{1}{2}$. Hence, $\mathbb{E}[S] = \frac{1}{2}\left(2F + \frac{F}{2}\right) = 1.25F$, and you are better off exchanging your selected (and still sealed) envelope with the the other sealed envelope. The same logic is applicable after the exchange, and swapping should continue ad infinitum.

(b) You are now allowed to pick one envelope and see how much cash is inside, and then based on this information, you can decide to switch envelopes or stick with the envelope you already have.

Can you come up with a strategy which will allow you to pick the envelope with more money, with probability strictly greater than $1/2$?