

1 Balls in Bins

You are throwing k balls into n bins. Let X_i be the number of balls thrown into bin i .

(a) What is $\mathbb{E}[X_i]$?

(b) What is the expected number of empty bins?

(c) Define a collision to occur when a ball lands in a nonempty bin (if there are n balls in a bin, count that as $n - 1$ collisions). What is the expected number of collisions?

2 Beast Arcade

One day you find yourself inside the Mr. Beast Arcade, which is full of games that pay YOU to play them!

- (a) In the first game, Chandler hands you a crisp \$20 bill up front. Then, he flips a coin that shows heads with probability p repeatedly, stopping when a heads comes up for the first time. You receive an additional dollar for each flip. How much money will you get in expectation?
- (b) In the next game, Karl rolls a fair 6-sided die. He then calculates 2^x , where x is the result of that die and hands you that much money. What is the expected amount of money you'll receive?
- (c) For the last game, Jimmy makes your friend flip a fair coin 10,000 times in a row, keeping track of the number of heads that show up. He then hands you a briefcase filled with \$1,000 and says he will also pay you \$5 for each head that comes up. Let X be a random variable representing the number of heads your friend flips. Use it to come up with an expression for Y , a random variable representing the total amount of money you'll receive.
- (d) What is $\mathbb{E}[Y]$? What about $\mathbb{P}[Y = 26,000]$?

3 Variance

- (a) Let X be a random variable representing the outcome of the roll of one fair 6-sided die. What is $\text{Var}(X)$?
- (b) Let Z be a random variable representing the average of n rolls of a fair 6-sided die. What is $\text{Var}(Z)$?

4 Covariance

- (a) We have a bag of 5 red and 5 blue balls. We take two balls uniformly at random from the bag without replacement. Let X_1 and X_2 be indicator random variables for the events of the first and second ball being red, respectively. What is $\text{cov}(X_1, X_2)$? Recall that $\text{cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$.

- (b) Now, we have two bags A and B, with 5 red and 5 blue balls each. Draw a ball uniformly at random from A, record its color, and then place it in B. Then draw a ball uniformly at random from B and record its color. Let X_1 and X_2 be indicator random variables for the events of the first and second draws being red, respectively. What is $\text{cov}(X_1, X_2)$?