1 Head Count II

Consider a coin with $\mathbb{P}[\text{Heads}] = 3/4$. Suppose you flip the coin until you see heads for the first time, and define X to be the number of times you flipped the coin.

(a) What is $\mathbb{P}[X = k]$, for some $k \geq 1$?

(b) Name the distribution of X and what its parameters are.

(c) What is $\mathbb{P}[X \geq k]$, for some $k \geq 1$?

(d) What is $\mathbb{P}[X \leq k]$, for some $k \geq 1$?

(e) What is $\mathbb{P}[X \geq k \mid X \geq m]$, for some $k \geq m \geq 1$? How does this relate to $\mathbb{P}[X \geq k - (m-1)]$?

2 Family Planning

Mr. and Mrs. Brown decide to continue having children until they either have their first girl or until they have three children. Assume that each child is equally likely to be a boy or a girl, independent of all other children, and that there are no multiple births. Let G denote the numbers of girls that the Browns have. Let C be the total number of children they have.

(a) Determine the sample space, along with the probability of each sample point.

(b) Compute the joint distribution of G and C. Fill in the table below.

	$C = 1$	$C = 2$	$C = 3$
$G = 0$			
$G = 1$			

(c) Use the joint distribution to compute the marginal distributions of G and C and confirm that the values are as you'd expect. Fill in the tables below.

$\mathbb{P}[G = 0]$	
$\mathbb{P}[G = 1]$	

$\mathbb{P}[C = 1]$	$\mathbb{P}[C = 2]$	$\mathbb{P}[C = 3]$

(d) Are G and C independent?

(e) What is the expected number of girls the Browns will have? What is the expected number of children that the Browns will have?

3 Pullout Balls

Suppose you have a bag containing four balls numbered $1, 2, 3, 4$.

(a) You perform the following experiment: pull out a single ball and record its number. What is the expected value of the number that you record?

(b) You repeat the experiment from part (a), except this time you pull out two balls together and record the product of their numbers. What is the expected value of the total that you record?

4 Double Counting

In this problem, we will show that linearity of expectation can be seen as a form of double counting.

Suppose Alice has two red and two blue marbles, where marbles of the same color are indistinguishable. She places the four marbles uniformly at random in a row. Let X be the random variable representing the number of times that a red and blue marble are adjacent.

(a) Fill in the following table:

Arrangement of marbles	1st and 2nd are RB or BR?	2nd and 3rd are RB or BR?	3rd and 4th are RB or BR?	Total number of RB or BR
RRBB				
RBRB				
BRRB				
RBBR				
BRBR				
BBRR				

(b) Without using linearity of expectation, compute $\mathbb{E}[X]$. (Hint: The table in part (a) will be helpful.)

(c) Now, let I_1, I_2, and I_3 be indicator variables, where I_i is 1 if the ith and $i+1$th marble are an adjacent red-blue or blue-red pair, and 0 if not. Without using the table in part (a), compute $\mathbb{E}[I_1]$.

(d) Using linearity of expectation, compute $\mathbb{E}[X]$.

(e) Argue why the different approaches in part (b) and part (d) should give the same answer. (Hint: Each of the expected values you computed corresponds to a different column of the table in part (a).)