CS 70 Fall 2022	Discrete Mathematics and Probability Theory Satish Rao and Babak Ayazifar	DIS 1A

1 Contraposition

Prove the statement "if $a+b < c+d$, then $a < c$ or $b < d$".

2 Numbers of Friends

Prove that if there are $n \geq 2$ people at a party, then at least 2 of them have the same number of friends at the party. Assume that friendships are always reciprocated: that is, if Alice is friends with Bob, then Bob is also friends with Alice.
(Hint: The Pigeonhole Principle states that if n items are placed in m containers, where $n > m$, at least one container must contain more than one item. You may use this without proof.)

3 Pebbles

Suppose you have a rectangular array of pebbles, where each pebble is either red or blue. Suppose that for every way of choosing one pebble from each column, there exists a red pebble among the chosen ones. Prove that there must exist an all-red column.

4 Preserving Set Operations

For a function f, define the image of a set X to be the set $f(X) = \{y \mid y = f(x) \text{ for some } x \in X\}$. Define the inverse image or preimage of a set Y to be the set $f^{-1}(Y) = \{x \mid f(x) \in Y\}$. Prove the following statements, in which A and B are sets.

Recall: For sets X and Y, $X = Y$ if and only if $X \subseteq Y$ and $Y \subseteq X$. To prove that $X \subseteq Y$, it is sufficient to show that $(\forall x)\ ((x \in X) \implies (x \in Y))$.

(a) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

(b) $f(A \cup B) = f(A) \cup f(B)$.